

# Orbit Spaces in Superconductivity

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## Abstract

In the framework of Landau theory of phase transitions one is interested to describe all the possible low symmetry “superconducting” phases allowed for a given superconductor crystal and to determine the conditions under which this crystal undergoes a phase transition. These problems are best described and analyzed in the orbit space of the high symmetry group of the “normal, non-superconducting” phase of the crystal. In this article it is worked out a simple example concerning superconductivity, that shows the  $\widehat{P}$ -matrix method to determine the equations and inequalities defining the orbit space and its stratification. This approach is of general validity and can be used in all physical problems that make use of invariant functions, as long as the symmetry group is compact. <sup>1</sup>

## 1 Introduction

Landau theory of phase transitions [1, 2, 3] has been sometimes used to study superconductor crystals. [4, 5] In that framework one deals with a given potential function  $V(x)$ , defined in a finite dimensional vector space  $X$ , that is invariant under transformations of a compact group  $G$  acting linearly on  $X$ :  $V(gx) = V(x)$ ,  $\forall g \in G$ ,  $x \in X$ . The vectors  $x \in X$  are called *order parameters*. All finite dimensional representations of a compact group  $G$  are unitary and completely reducible, and, given a complex irreducible representation  $\varphi$  of  $G$ , either  $\varphi$  itself, or the direct sum  $\varphi + \bar{\varphi}$ , where  $\bar{\varphi}$  is the complex conjugate representation of  $\varphi$ , is equivalent to a real orthogonal irreducible (on the reals) representation of  $G$ . [6, 7] In Landau theory, in a first approximation, sufficient to determine all possible symmetry phases allowed for the system, one considers  $X$  hosting a representation of  $G$  equivalent to a real orthogonal irreducible representation. [2] This implies that no vector  $x \neq 0$  is left invariant by all elements of  $G$ , so the action is *effective*. In all generality, one may then assume that  $X \cong \mathbb{R}^n$  and that the group  $G$  is a real orthogonal subgroup of  $O(n, \mathbb{R})$ . In the following the irreducibility condition will never be used, but we still

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require effective orthogonal actions in  $\mathbb{R}^n$ .

In this Introduction some of the mathematical tools developed in [8, 9] are reviewed; they are used in Section 2 to determine the symmetry phases allowed for the physical system considered. The method exposed modifies slightly that one proposed by Gufan. [10] The physical results here obtained are not new, [11] but I would instead point the attention of the reader to the mathematical method used to derive the orbit space stratification and the list of phase transitions allowed for a given potential.

If  $x_0$  is the minimum point of  $V(x)$ , then the system is in equilibrium in a state compatible with the order parameters  $x_0$ . If  $x_0 \neq 0$  then the symmetry of the system is lower than  $G$ , it is reduced to  $G_{x_0} = \{g \in G \mid gx_0 = x_0\}$ , the *isotropy subgroup* of  $x_0$ , realizing a *spontaneous symmetry breaking*.

Moreover, if  $x_0 \neq 0$ , the *orbit*  $\Omega(x_0) = \{gx_0, g \in G\}$  contains points  $x \neq x_0$  where  $V(x) = V(x_0)$ , so the low symmetry equilibrium states are always degenerate. The following relation holds: [12, 3]  $G_{gx} = gG_xg^{-1}, \forall g \in G, x \in \mathbb{R}^n$ , then all points  $x' \in \Omega(x)$  have isotropy subgroups in the same conjugacy class  $[G_x]$  and viceversa, all groups  $G' \in [G_x]$  are isotropy subgroups of some points  $x' \in \Omega(x)$ . The conjugacy class  $[G_x]$  is called the *(isotropy) type* of  $\Omega(x)$ . Two conjugated isotropy subgroups are physically equivalent because they can be thought to be the same group seen from two rotated systems of reference, of course we are here considering  $G$  orthogonal. Given an isotropy subgroup  $H \subseteq G$ , the set of points  $\Sigma_{[H]} = \{x \in \mathbb{R}^n \mid G_x \in [H]\}$  is called the *stratum* of type  $[H]$ . The strata are disjoint and in a one to one correspondence with the *symmetry phases* that may in principle be accessible to the system. Orbit and strata can be partially ordered according to their types: the type  $[H]$  is said to be *smaller* than the type  $[K]$ ,  $[H] < [K]$ , if  $H' \subset K'$  for some  $H' \in [H]$  and  $K' \in [K]$ . If  $G$  is compact and is acting orthogonally in  $\mathbb{R}^n$ , the number of strata is finite, [13] p. 444, and there is a unique stratum of the smallest type, [12] called the *principal stratum*.

The values assumed by the potential  $V$  generally depend, in addition to  $x$ , on other variables, like temperature, pressure, atomic concentrations, etc., and the absolute minimum of  $V$  falls in a point  $x_0$  that depends on these variables. Sometimes, when these variables change their values,  $x_0$  may change stratum, and in this case the system undergoes a *phase transition*.

The *orbit space* of  $G$  is the quotient space  $\mathbb{R}^n/G$ , whose points represent whole orbits of  $\mathbb{R}^n$ . A stratum of the orbit space is formed by all the points that represent orbits in a same stratum of  $\mathbb{R}^n$ . The invariant functions, e.g.  $V(x)$ , are naturally defined in the orbit space because they are constant on the orbits, so, to eliminate the degeneracy associated to the symmetry, one is lead to study the invariant functions in the orbit space.

To determine concretely the equations and inequalities that define the orbit space and its strata one may take advantage of the  $\widehat{P}$ -matrix and its properties. [14, 8] By Hilbert's theorem, [15, 16] the algebra  $\mathbb{R}[\mathbb{R}^n]^G$  of real  $G$ -invariant polynomials defined in  $\mathbb{R}^n$  has a finite number  $q$  of generators:  $p_1, \dots, p_q \in \mathbb{R}[\mathbb{R}^n]^G$ , forming an *integrity basis* for  $\mathbb{R}[\mathbb{R}^n]^G$ .

This means that

$$f(x) = \widehat{f}(p_1(x), \dots, p_q(x)) \quad \forall f \in \mathbb{R}[\mathbb{R}^n]^G, \quad x \in \mathbb{R}^n,$$

with  $\widehat{f}$  a real polynomial in  $q$  variables. Actually, the preceding formula holds true for any  $G$ -invariant  $C^\infty$  function  $f(x)$ , then  $\widehat{f}$  is a  $C^\infty$  function of  $q$  variables. [17, 18] If no subset of  $\{p_1, \dots, p_q\}$  is still an integrity basis for  $\mathbb{R}[\mathbb{R}^n]^G$ , then the set  $\{p_1, \dots, p_q\}$  forms a *minimal integrity basis* (MIB) for  $\mathbb{R}[\mathbb{R}^n]^G$ . There is some arbitrariness in the choice of the MIB, however all its elements can be chosen to be homogeneous and one may order the  $p_a$ ,  $a = 1, \dots, q$ , according to their degrees  $d_a = \deg p_a(x)$ , for example  $d_a \geq d_{a+1}$ . If  $G$  acts effectively in  $\mathbb{R}^n$ , then  $d_q = 2$ , and, for the orthogonality of  $G$ , one may take  $p_q(x) = |x|^2 = \sum_{i=1}^n x_i^2$ .

The vector map  $p : \mathbb{R}^n \rightarrow \mathbb{R}^q : x \rightarrow p(x) = (p_1(x), p_2(x), \dots, p_q(x))$ , called the *orbit map*, is constant in each orbit  $\Omega \subset \mathbb{R}^n$  because the  $p_a(x)$  are  $G$ -invariant functions. The point  $p = p(x) \in \mathbb{R}^q$  is the image in  $\mathbb{R}^q$  of the orbit  $\Omega(x)$  of  $\mathbb{R}^n$ . No other orbit  $\Omega' \neq \Omega$  of  $\mathbb{R}^n$  is mapped to the same point  $p \in \mathbb{R}^q$  because the MIB separates the orbits. [18, 8] The image of  $\mathbb{R}^n$  through the orbit map is the set

$$\mathcal{S} = p(\mathbb{R}^n) \subset \mathbb{R}^q$$

that can be identified with the orbit space of the  $G$ -action. [18]

The strata  $\sigma$  of  $\mathcal{S}$  are the images of the strata  $\Sigma$  of  $\mathbb{R}^n$  through the orbit map:  $\sigma = p(\Sigma)$ . The principal stratum  $\sigma_p$  of  $\mathcal{S}$  is connected, open and dense in  $\mathcal{S}$ . [19] Given two strata  $\Sigma$  and  $\Sigma'$  of  $\mathbb{R}^n$ , if  $\Sigma'$  is of greater type than  $\Sigma$ , then  $\sigma' = p(\Sigma')$  lie in the boundary of  $\sigma = p(\Sigma)$ . [20] The smallest stratum of  $\mathcal{S}$  is  $\sigma_{[G]}$ , image of the origin of  $\mathbb{R}^n$  and is located at the origin of  $\mathbb{R}^q$ . Given any plane  $\Pi_r$  of  $\mathbb{R}^q$  with equation  $p_q = r^2 > 0$ ,  $\mathcal{S} \cap \Pi_r$ , is a *non-empty compact connected* section of  $\mathcal{S}$  that contains all strata of  $\mathcal{S}$  except  $\sigma_{[G]}$ . [8, 9] This section is sufficient to describe the whole shape of  $\mathcal{S}$ , because moving  $\Pi_r$  toward the infinite or toward the origin of  $\mathbb{R}^q$ , by changing  $r^2$ ,  $\mathcal{S} \cap \Pi_r$  expands or contracts, but maintains its topological shape and stratification.

Let's define the  $q \times q$  symmetric and positive semi-definite matrix  $P(x)$ , with elements

$$P_{ab}(x) = \nabla p_a(x) \cdot \nabla p_b(x) = \sum_{i=1}^n \frac{\partial p_a(x)}{\partial x_i} \frac{\partial p_b(x)}{\partial x_i} \quad a, b = 1, \dots, q.$$

The matrix elements  $P_{ab}(x)$  are real homogeneous polynomial functions of  $x$  with  $\deg P_{ab}(x) = d_a + d_b - 2$ , and  $P_{qa}(x) = P_{aq}(x) = 2d_a p_a(x)$ . A less immediate property is the  $G$ -invariance; it follows from the orthogonality of  $G$  and from the covariance of the gradients of  $G$ -invariant functions:

$$P_{ab}(gx) = \nabla p_a(gx) \cdot \nabla p_b(gx) = g \nabla p_a(x) \cdot g \nabla p_b(x) = \nabla p_a(x) \cdot \nabla p_b(x) = P_{ab}(x).$$

Then,  $P_{ab}(x)$  can be expressed in terms of the MIB:

$$P_{ab}(x) = \widehat{P}_{ab}(p_1(x), \dots, p_q(x)), \quad \forall x \in \mathbb{R}^n, \quad a, b = 1, \dots, q.$$

The matrix  $\widehat{P}(p)$ , defined in  $\mathbb{R}^q$ , is called the  $\widehat{P}$ -matrix. When  $p = p(x)$ , then  $\widehat{P}(p)$  coincides with  $P(x)$ .

Let's call a polynomial  $\widehat{f}(p)$  *w-homogeneous of weight d* if the polynomial  $f(x) = \widehat{f}(p(x))$  is homogeneous of degree  $d$ , and define the *surface of the relations*:  $\mathcal{Z} = \{p \in \mathbb{R}^q \mid \widehat{f}_i(p) = 0, i = 1, \dots, k\}$ , where  $\widehat{f}_i(p_1(x), \dots, p_q(x)) = 0, i = 1, \dots, k$ , are  $k$  algebraically independent homogeneous polynomial relations, called *syzygies*, existing among the  $p_a(x)$ . If there are no syzygies, then  $k = 0$  and  $\mathcal{Z} \equiv \mathbb{R}^q$ .

The main properties of a  $\widehat{P}$ -matrix are the following: [8, 21]

- $\widehat{P}(p)$  is a real, symmetric  $q \times q$  matrix. The matrix elements  $\widehat{P}_{ab}(p)$  are *w-homogeneous polynomials of the  $p_a$  of weight  $d_a + d_b - 2$* , and  $\widehat{P}_{qa}(p) = \widehat{P}_{aq}(p) = 2d_a p_a, \forall a = 1, \dots, q$ .
- $\widehat{P}(p)$  is positive semi-definite in  $\mathcal{S}$ . Precisely,  $\mathcal{S} = \{p \in \mathcal{Z} \mid \widehat{P}(p) \geq 0\}$ ;
- Given a stratum  $\sigma \subset \mathcal{S}$ , and  $p \in \sigma$ ,  $\dim \sigma = \text{rank } \widehat{P}(p)$ .

The  $\widehat{P}$ -matrix completely determines the orbit space  $\mathcal{S}$  and its stratification. Defining  $\mathcal{S}_d$  as the union of all  $d$ -dimensional strata of  $\mathcal{S}$ , one has:

$$\mathcal{S}_d = \{p \in \mathcal{Z} \mid \widehat{P}(p) \geq 0, \text{rank } \widehat{P}(p) = d\},$$

so,  $\mathcal{S} = \overline{\mathcal{S}_{q-k}}$ , where  $q - k = \dim \mathcal{Z}$  and  $k$  is the number of syzygies. As all these positivity and rank conditions are expressed through a finite number of polynomial equations and inequalities,  $\mathcal{S}$  is a semi-algebraic set with a finite number of strata (confirming Mostow result [13]).

The polynomials defining the strata  $\sigma$  of  $\mathcal{S}$  of dimension  $q - 1$  (that exist only if there is at most one syzygy) must satisfy a set of differential relations that characterize them. [22, 9] Let  $a(p)$  be an irreducible polynomial defining such a stratum of  $\mathcal{S}$ . Then the following *master* relations hold true: [22]

$$\sum_{b=1}^q \widehat{P}_{ab}(p) \frac{\partial a(p)}{\partial p_b} = \lambda_a(p) a(p) \quad a = 1, \dots, q,$$

where the  $\lambda_a(p)$  are *w-homogeneous polynomials of weight  $d_a - 2$* . The polynomials  $a(p)$  that satisfy the above master relations are called *active*. They are always factors of  $\det \widehat{P}(p)$ . [9] The  $(q - 1)$ -dimensional stratum  $\sigma$  is located in the interior of  $\overline{\sigma} = \{p \in \mathbb{R}^q \mid a(p) = 0, \widehat{P}(p) \geq 0\}$ . The border  $\overline{\sigma} \setminus \sigma$  of  $\sigma$  is the union of lower dimensional strata whose defining equations and inequalities can be found simplifying the system  $\{a_1(p) = a_2(p) = 0, \widehat{P}(p) \geq 0\}$ , where  $a_1(p)$  and  $a_2(p)$  are two irreducible active polynomials.

As the  $\widehat{P}$ -matrices completely define the orbit spaces and their stratification, the classification of  $\widehat{P}$ -matrices implies the classification of orbit spaces. All  $\widehat{P}$ -matrices of dimension  $q \leq 4$  corresponding to *coregular* groups (i.e. with no syzygies) have been determined and classified. [9, 23] This classification is obtained independently from the specific group structures by making use only of some general properties of orbit spaces. It is found that many different groups, no matter if they are finite groups or compact Lie groups, have the same  $\widehat{P}$ -matrix and the same orbit space. [24]

## 2 A concrete example analyzed in the orbit space

The example here proposed was first studied by Gufan [11] and concerns the study of the possible symmetry phases of electron pairs with non-zero total orbital momentum in hexagonal superconducting crystals, in the approximation of strong crystalline field and strong spin-orbit interaction. In [11] the order parameters are chosen to be two complex numbers,  $\eta_+$  and  $\eta_-$ , and the symmetry group is  $\widehat{Y} = \widehat{G} \otimes \widehat{R} \otimes \widehat{U}_1(\alpha)$ , where  $\widehat{G}$  is the point symmetry group of the crystal,  $\widehat{R}$  is the time reversal group and  $U_1(\alpha)$  is the rotation group of the complex plane. Here we consider only the case  $\widehat{G} = C_{3v}$ . The group  $\widehat{Y}$  acts in the four dimensional complex space of vectors of the following form:

$$\eta = \begin{pmatrix} \eta_+ \\ \eta_- \\ \eta_-^* \\ \eta_+^* \end{pmatrix},$$

where  $^*$  denotes complex conjugation, and the group generators have the following matrix expressions:

$$\begin{pmatrix} C_3^{1z} \\ e^{2\pi i/3} \\ e^{-2\pi i/3} \\ e^{2\pi i/3} \\ e^{-2\pi i/3} \end{pmatrix} \begin{pmatrix} \sigma^v \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} U_1(\alpha) \\ e^{i\alpha} \\ e^{i\alpha} \\ e^{-i\alpha} \\ e^{-i\alpha} \end{pmatrix} \begin{pmatrix} R \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where  $C_3^{1z}$  and  $U_1(\alpha)$  are diagonal matrices, of which only the diagonal elements are shown.

The MIB for this group has 3 elements that can be written in the following real form:

$$J_3 = |\eta_+|^2 + |\eta_-|^2, \quad J_2 = |\eta_+|^2 |\eta_-|^2, \quad J_1 = \eta_+^3 \eta_-^* + \eta_+^* \eta_-^3.$$

Of course one may consider the action of  $\widehat{Y}$  in  $\mathbb{C}^2$ , but this action would not be linear, the invariants would not be polynomial functions of  $\eta_+$  and  $\eta_-$ , and the theory exposed in the Introduction would not be valid.

When a unitary representation of a compact group is equivalent to a real orthogonal representation, one may always choose the MIB's for the two cases (unitary and orthogonal) to be formed by the same real polynomial functions. [24] This means that the orbit map defines in the two cases the same subset  $\mathcal{S} \subset \mathbb{R}^q$ , that is the same orbit space.

To adapt this scheme to that one reviewed in the Introduction, one has to consider an orthogonal action of  $\widehat{Y}$  in the real space  $\mathbb{R}^4$ . Introducing 4 real variables  $x_{1,2}$  and  $y_{1,2}$ , one may put  $\eta_+ = x_1 + iy_1$  and  $\eta_- = x_2 + iy_2$ , and consider a vector  $x \in \mathbb{R}^4$  of the following form:

$$x = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}.$$

This defines a one-to-one correspondence between the complex space of the  $\eta$  and  $\mathbb{R}^4$ . The group  $\widehat{Y}$  acts in  $\mathbb{R}^4$  with the following matrix generators:

$$\begin{array}{cc} C_3^{1z} & \sigma^v \\ \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ U_1(\alpha) & R \\ \begin{pmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \end{array}$$

where  $c = \cos \alpha$  and  $s = \sin \alpha$ .

The MIB can be easily written down in terms of the real variables:

$$\begin{aligned} p_3(x) &= x_1^2 + x_2^2 + y_1^2 + y_2^2, \\ p_2(x) &= 4(x_1^2 + y_1^2)(x_2^2 + y_2^2), \\ p_1(x) &= 16(x_1 x_2 + y_1 y_2)(x_1^2 x_2^2 - 3x_1^2 y_2^2 + 8x_1 x_2 y_1 y_2 - 3x_2^2 y_1^2 + y_1^2 y_2^2), \end{aligned}$$

where the numeric factors have been introduced for later convenience.

To find out the  $\widehat{P}$ -matrix one has to calculate all gradients of the  $p_a(x)$ , make their scalar products, and express these products in terms of the  $p_a(x)$ . One finds, at the end, the following  $\widehat{P}$ -matrix:

$$\widehat{P}(p) = \begin{pmatrix} 144 p_2^2 p_3 & 24 p_1 p_3 & 12 p_1 \\ 24 p_1 p_3 & 16 p_2 p_3 & 8 p_2 \\ 12 p_1 & 8 p_2 & 4 p_3 \end{pmatrix},$$

whose determinant is

$$\det \widehat{P}(p) = 2304 p_3 (p_1^2 - 4p_2^3)(p_2 - p_3^2).$$

The  $p_a(x)$  are algebraically independent, so the principal stratum is the subset of  $\mathbb{R}^3$  where  $\widehat{P}(p) > 0$ , and all other strata are contained in the surface of equation  $A(p) = 0$ , where  $A(p)$  is the product of all the irreducible active factors of  $\det \widehat{P}(p)$ . Using the master relations, one finds:

$$A(p) = (p_1^2 - 4p_2^3)(p_2 - p_3^2).$$

The section of the surface  $A(p) = 0$  in a plane  $p_3 = \text{constant}$  is plotted in Figure 1. Its interior hosts the section of the principal strata  $\sigma_p$  and its border contains the sections of all other strata, except the origin. The complete list of the strata of  $\mathcal{S}$  is reported in Table 1, where the defining equations and inequalities of the bordering strata are found by examining  $\text{rank } \widehat{P}(p)$  in the surface  $A(p) = 0$ .

Table 2 lists for each stratum  $\sigma$ , some conveniently chosen points  $\bar{x}$  and the generators of their isotropy subgroups  $G_{\bar{x}}$ . One may note that the points in the border of a stratum  $\sigma$  have isotropy subgroups containing those of points of  $\sigma$ , modulo a conjugation. Just to see this fact, in two

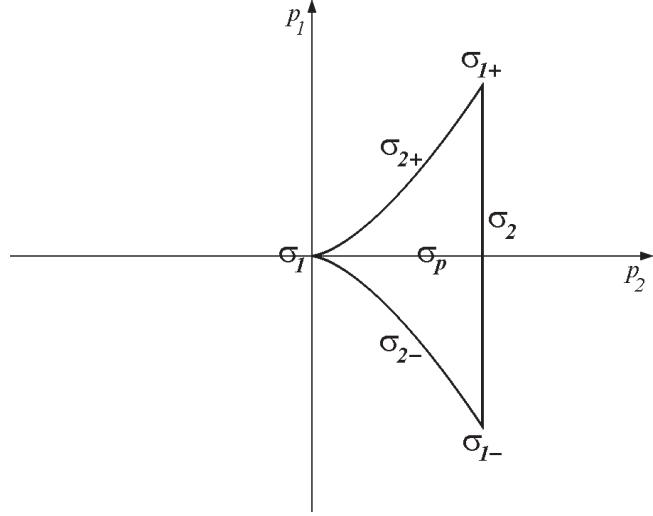


Figure 1: Section of the orbit space.

Table 1: Strata of the orbit space

Stratum	Defining relations
$\sigma_0$	$p_1 = p_2 = p_3 = 0$
$\sigma_1$	$p_1 = p_2 = 0, p_3 > 0$
$\sigma_{1+}$	$p_1 = 2p_3^3, p_2 = p_3^2, p_3 > 0$
$\sigma_{1-}$	$p_1 = -2p_3^3, p_2 = p_3^2, p_3 > 0$
$\sigma_2$	$ p_1  < 2p_3^3, p_2 = p_3^2, p_3 > 0$
$\sigma_{2+}$	$p_1 = 2p_2^{\frac{3}{2}}, 0 < p_2 < p_3^2, p_3 > 0$
$\sigma_{2-}$	$p_1 = -2p_2^{\frac{3}{2}}, 0 < p_2 < p_3^2, p_3 > 0$
$\sigma_p$	$p_1^2 < 4p_2^3, 0 < p_2 < p_3^2, p_3 > 0$

cases are reported two different points in a same orbit, related by the group element  $U_1(\frac{\pi}{2})$ .

Once the orbit space is described, one may list all second order phase transitions that may in principle be allowed for the system, a second order phase transition may in fact take place only between bordering strata because it is a consequence of a continuous variation of the variables that appear in the potential. [1, 2, 3] In the example here considered, the possible second order phase transitions are the following ones:

$$\begin{array}{lll} \sigma_0, \sigma_p & \longleftrightarrow & \text{all other strata} \\ \sigma_2 & \longleftrightarrow & \sigma_{1+}, \sigma_{1-} \end{array} \quad \begin{array}{lll} \sigma_{2+} & \longleftrightarrow & \sigma_1, \sigma_{1+} \\ \sigma_{2-} & \longleftrightarrow & \sigma_1, \sigma_{1-} \end{array}$$

The conditions under which the phase transitions are realized depend on the form assumed for the potential when this is written in terms of the MIB and not all the transitions listed above do actually take place when

Table 2: Isotropy types of the strata

Stratum	Typical point $\bar{x}$	Generators of the isotropy subgroup of $\bar{x}$
$\sigma_0$	(0,0,0,0)	$U_1(\alpha); C_3^{1z}; \sigma^v; R$
$\sigma_1$	(0,1,0,0)	$U_1(-\frac{2}{3}\pi)C_3^{1z}; U_1(\pi)\sigma^v R$
$\sigma_{1+}$	(0,1,0,1)	$\sigma^v; U_1(\pi)R$
$\sigma_{1-}$	(-1,0,1,0)	$U_1(\pi)\sigma^v; U_1(\pi)R$
	(0,1,0,-1)	$U_1(\pi)\sigma^v; R$
$\sigma_2$	(-1,1,1,1)	$U_1(\pi)R$
	(1,1,1,-1)	$R$
$\sigma_{2+}$	(0, 1, 0, 2)	$U_1(\pi)\sigma^v R$
$\sigma_{2-}$	(0, 1, 0, -2)	$U_1(\pi)\sigma^v R$
$\sigma_p$	(-1, 2, 1, 1)	$E$

a given form of the potential is specified. It is not possible to examine here in detail the phase transitions scenario associated to a given potential  $\hat{V}(p)$  and I shall here only remind the basic points of this analysis. [24, 11] One first has to write down  $\hat{V}(p)$  in terms of the MIB. Generally, one uses a polynomial approximation for  $\hat{V}(p)$ :  $\hat{V}(p) = a_1 p_3 + a_2 p_3^2 + \dots + b_1 p_2 + \dots + c_1 p_1 + \dots$ , truncated to a certain degree. One determines then the conditions on the parameters  $a_i, b_i, \dots$  in order that  $\hat{V}(p)$  be bounded below in each of the strata of  $\mathcal{S}$ , determines these constrained minima, and compares them to single out the absolute minimum. At the end one knows, for any set of values for the parameters, which is the stable phase, that is the stratum hosting the absolute minimum of  $\hat{V}(p)$ . It is evident that polynomial potentials of small degrees cannot develop minima in certain strata of  $\mathcal{S}$  for any set of values of the parameters.

The order parameters are linked to some physical observables. This is the case, for instance, of the number of pairs  $n = |\eta_+|^2 + |\eta_-|^2 = p_3$ , of the number  $z = |\eta_+|^2 - |\eta_-|^2 = \pm\sqrt{p_3^2 - p_2}$  that changes sign under time reversal and determines the magnetic properties of the crystal, [2] and of the components of the deformation tensor of the unit cell of the crystal. From Table 1 one sees that in the strata  $\sigma_0, \sigma_{1-}, \sigma_{1+}, \sigma_2$ , and only in them,  $z$  vanishes identically, so these strata cannot exhibit magnetic properties. The change of the sign of  $p_1$  does not modify the  $\hat{P}$ -matrix and the orbit space  $\mathcal{S}$ , but changes the strata  $\sigma_{1-} \leftrightarrow \sigma_{1+}$  and  $\sigma_{2-} \leftrightarrow \sigma_{2+}$ . As this sign change has no physical meaning, the strata  $\sigma_{1-}$  and  $\sigma_{2-}$  are not physically distinguishable from  $\sigma_{1+}$  and  $\sigma_{2+}$ , nevertheless they are distinct strata lying in different positions of the orbit space, and phase transitions concerning each one of these strata are possible.

The  $\hat{P}$ -matrix and the orbit space here described, also appears in the case of superconducting cubic crystals, if the order parameters are assumed to transform according to the two-dimensional complex representation of the cubic symmetry group. [25] In that case however the isotropy groups are no longer those reported in Table 2. As already noted, many different symmetry groups may give rise to the same  $\hat{P}$ -matrix and the same orbit space. When this happens, the number of different symmetry

phases and the list of the phase transitions allowed for these systems are the same (but the corresponding isotropy groups are generally different, depending on the specific high symmetry group considered). Moreover, for a given form of the potential in terms of the MIB, the conditions to realize the listed phase transitions impose the same conditions on the parameters of the potential.

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